On a question of A. Balog *

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Annotation.

We give a partial answer to a conjecture of A. Balog, concerning the size of $AA + A$, where A is a finite subset of real numbers. Also, we prove several new results on the cardinality of $A : A + A$, $AA + AA$ and $A : A + A : A$.

1 Introduction

Let $A \subset \mathbb{R}$ be a finite set. Define the *sumset*, and respectively the *product set*, by

$$A + A := \{a + b : a, b \in A\}$$

and

$$AA := \{ab : a, b \in A\}.$$

The Erdős-Szemerédi [2] conjecture states, for all $\epsilon > 0$,

$$\max\{|A + A|, |AA|\} \gg |A|^{2-\epsilon}\,.$$

Loosely speaking, the conjecture says that any set of reals (or integers) cannot be highly structured both in a multiplicative and additive sense. The best result in the direction is due to Solymosi [17].

Theorem 1 *Let $A \subset \mathbb{R}$ be a set. Then*

$$\max\{|A + A|, |AA|\} \gg |A|^{\frac{4}{3}} \log^{-\frac{1}{3}} |A|\,.$$

If one consider the set

$$AA + A = \{ab + c \ : \ a, b, c \in A\}$$

then the Erdős-Szemerédi conjecture implies that $AA + A$ has size at least $|A|^{2-\varepsilon}$ (we assume for simplicity that $1 \in A$). In [1] Balog formulated a weaker hypothesis that for all $\varepsilon > 0$ one has

$$|AA + A| \gg |A|^{2-\varepsilon}\,.$$

In the paper he proved the following result, which implies, in particular, $|AA + A| \gg |A|^{3/2}$ and $|AA + AA| \gg |A||A/A|^{1/2}$.

*This work was supported by grant Russian Scientific Foundation RSF 14–11–00433.

Theorem 2 *For every finite sets of reals $A, B, C, D \subset \mathbb{R}$, we have*

$$|AC + A||BC + B| \gg |A||B||C|\,, \tag{1}$$

and

$$|AC + AD||BC + BD| \gg |B/A||C||D|\,. \tag{2}$$

More precisely, see [11]

$$|(A \times B) \cdot \Delta(C) + A \times B| \gg |A||B||C|$$

and

$$|(A \times B) \cdot \Delta(C) + (A \times B) \cdot \Delta(D)| \gg |B/A||C||D|\,,$$

where

$$\Delta(A) := \{(a, a) \ : \ a \in A\}\,.$$

In [7] the authors have obtained a partial answer to a "dual" question on the size of $A(A+A)$. The main result of the paper is the following new bound for $A : A + A$, $AA + A$, more precisely, see Theorem 13 below.

Theorem 3 *Let $A \subset \mathbb{R}$ be a set. Then there is $\varepsilon_1 > 0$ such that*

$$|A : A + A| \gg |A|^{3/2+\varepsilon_1}\,. \tag{3}$$

Moreover, there is $\varepsilon_2 > 0$ with

$$|AA + A| \gg |A|^{3/2+\varepsilon_2}\,, \tag{4}$$

provided by $|A : A| \ll |AA|$.

Also, we prove several results on the cardinality of $AA + AA$ and $A : A + A : A$, see Theorem 15 and Proposition 16 below.

In paper [9] Roche–Newton and Zhelezov conjectured there exist absolute constants c, c' such that for any finite $A \subset \mathbb{C}$ the following holds

$$\left|\frac{A+A}{A+A}\right| \le c|A|^2 \implies |A + A| \le c'|A|\,.$$

Similar conjectures were made for the sets $\frac{A-A}{A-A}$, $(A - A)(A - A)$, $A(A + A + A + A)$ and so on. We finish the paper giving a partial answer to a variant of the conjecture of Roche–Newton and Zhelezov

$$|(A + A)(A + A) + (A + A)(A + A)| \ll |A|^2 \implies |A \pm A| \ll |A| \log |A|\,,$$

see Corollary 18.

The main idea of the proof is the following. We need to estimate from below the sumset of two sets A and $A : A$, say. As in many problems of the type usual applications of Szemerédi–Trotter's theorem [18] or Solymosi's method [1] give us a lower bound of the form $|A : A + A| \gg$

$|A|^{3/2}$. In paper [12] the exponent 3/2 was improved in the particular case of sumsets of convex sets. After that the method was developed by several authors, see e.g. [4, 5, 6, 10, 11, 13, 14, 15] and others. In [15] the author proved that the bound $|A+B| \gg |A|^{3/2+c}$, $c > 0$ takes place for wide class of *different* sets, having roughly comparable sizes. For example, such bound holds if A and B have small multiplicative doubling. It turns out that if (3) cannot be improved then there is some large set C such that $|AC| \ll |A|$. This allows us to apply results from [15].

The author is grateful to Tomasz Schoen for useful discussions.

2 Notation

Let $\mathbf{G}$ be an abelian group and $+$ be the group operation. In the paper we use the same letter to denote a set $S \subseteq \mathbf{G}$ and its characteristic function $S : \mathbf{G} \to \{0, 1\}$. By $|S|$ denote the cardinality of S.

Let $f, g : \mathbf{G} \to \mathbb{C}$ be two functions with finite supports. Put

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y) g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y) g(y + x)\,. \tag{5}$$

Let $A \subseteq \mathbf{G}$ be a set. For any real $\alpha > 0$ put

$$\mathsf{E}^{+}_{\alpha}(A) = \sum_{x \in \mathbf{G}} (A \circ A)^{\alpha}(x) \tag{6}$$

be *the higher energy* of A. In particular case $k = 2$ we write $\mathsf{E}^{+}(A) = \mathsf{E}^{+}_{2}(A)$ and $\mathsf{E}(A, B)$ for $\sum_{x \in \mathbf{G}} (A \circ A)(x)(B \circ B)(x)$. The quantity $\mathsf{E}^{+}(A)$ is called *the additive energy* of a set, see e.g. [18]. For a sequence $s = (s_1, \dots, s_{k-1})$ put $A^{+}_{s} = A \cap (A - s_1) \cdots \cap (A - s_{k-1})$. Then

$$\mathsf{E}^{+}_{k}(A) = \sum_{s_1, \dots, s_{k-1} \in \mathbf{G}} |A^{+}_{s}|^2\,.$$

If we have a group $\mathbf{G}$ with a multiplication instead of addition then we use symbol $\mathsf{E}^{\times}_{\alpha}(A)$ for the correspondent energy of a set A and write $A^{\times}_{s}$ for $A \cap (As_1^{-1}) \cdots \cap (As_{k-1}^{-1})$. In the case of a unique operation we write just $\mathsf{E}_k(A)$, $\mathsf{E}(A)$ and A_s.

Let $A, B \subseteq \mathbf{G}$ be two finite sets. *The magnification ratio* $R_B[A]$ of the pair (A, B) (see e.g. [18]) is defined by

$$R_B[A] = \min_{\emptyset \neq Z \subseteq A} \frac{|B + Z|}{|Z|}\,. \tag{7}$$

A beautiful result on magnification ratio was proven by Petridis [8].

Theorem 4 *For any $A, B, C \subseteq \mathbf{G}$, we have*

$$|B + C + X| \le R_B[A] \cdot |C + X|\,, \tag{8}$$

where $X \subseteq A$ and $|B + X| = R_B[A]|X|$.

We conclude the section by Ruzsa's triangle inequality, see e.g. [18]. Interestingly, that our proof (developing some ideas of papers [11], [7]) does not require any mapping as usual.

Lemma 5 *Let $A, B, C \subseteq \mathbf{G}$ be any sets. Then*

$$|C||A-B| \le |A \times B - \Delta(C)| \le |A-C||B-C|\,. \tag{9}$$

P r o o f. We have

$$|A \times B - \Delta(C)| = \sum_{z \in A-B} |B \cap (A-z) - C| \ge |A-B||C|\,.$$

The inequality above is trivial and the identity follows by the projection of points $(x,y) \in A \times B - \Delta(C)$, $(x,y) = (a-c, b-c)$, $a \in A$, $b \in B$, $c \in C$ onto $z := x - y = a - b \in A - B$. This concludes the proof. $\Box$

All logarithms are base 2. Signs $\ll$ and $\gg$ are the usual Vinogradov's symbols.

3 Preliminaries

As we discussed in the introduction our proof uses some notions from [15]. So, let us recall the main definition of the paper.

Definition 6 *A set $A \subset \mathbf{G}$ has* **SzT–type** *(in other words A is called* **Szemerédi–Trotter set***) with parameter $\alpha \ge 1$ if for any set $B \subset \mathbf{G}$ and an arbitrary $\tau \ge 1$ one has*

$$|\{x \in A+B \ :\ (A*B)(x) \ge \tau\}| \ll c(A)|B|^\alpha \cdot \tau^{-3}\,, \tag{10}$$

where $c(A) > 0$ is a constant depends on the set A only.

Simple calculations (or see [15], Lemma 7) give us some connections between various energies of SzT–type sets. Formula (11) below is due to Li [5].

Lemma 7 *Suppose that $A, B, C \subseteq \mathbf{G}$ have SzT–type with the same parameter α. Then*

$$\mathsf{E}^3(A) \ll \mathsf{E}^2_{3/2}(A) c(A) |A|^\alpha\,, \tag{11}$$

$$\mathsf{E}(A) \ll c^{1/2}(A) |A|^{1+\alpha/2}\,, \tag{12}$$

and

$$\sum_x (A \circ A)(x)(B \circ B)(x)(C \circ C)(x) \ll (c(A)c(B)c(C))^{1/3} (|A||B||C|)^{\alpha/3} \times$$

$$\times \log(\min\{|A|, |B|, |C|\})\,. \tag{13}$$

We need in Lemma 27 from [11].

Lemma 8 *Any set $A \subset \mathbb{R}$, $\mathbb{R} = (\mathbb{R}, +)$ has SzT–type with $\alpha = 2$ and $c(A) = |A| d(A)$, where*

$$d(A) := \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|} \,. \tag{14}$$

So, any set with small multiplicative doubling or, more precisely, with small quantity (14) has SzT–type, relatively to addition, in an effective way. It can be checked that minimum in (14) is actually attained and we left the fact to an interested reader. Careful analysis of our proof gives that we do not need this. Another examples of SzT–types sets can be found in [15].

Now let us prove a simple result on $d(A)$, which follows from Petridis's Theorem 4.

Lemma 9 *Let $A \subseteq \mathbb{R}^+$ be a set. Then $d(A) = d(A^{-1})$ and*

$$d(AA) \le \frac{|A|^2 d^2(A)}{|AA||C|} \,, \qquad d(A:A) \le \frac{|A|^2 d^2(A)}{|A:A||C|} \,, \tag{15}$$

where C is a set where the minimum in (14) is attained.

P r o o f. The identity $d(A) = d(A^{-1})$ is obvious. Let us prove (15). Suppose that the minimum in (14) is attained at C. By Theorem 4 there is $X \subseteq C$ such that $|AAX| \le R|AX|$, where $R = R_A[C]$ is defined by formula (7). We have

$$d(AA) \le \frac{|AAX|^2}{|AA||X|} \le R^2 \frac{|AX|^2}{|AA||X|} = \frac{|AX|^4}{|AA||X|^3} \le \frac{|AC|^4}{|AA||C|^3} = \frac{d^2(A)|A|^2}{|AA||C|} \,. \tag{16}$$

Similarly, we take C such that the correspondent minimum for $d(A^{-1})$ is attained at C. Further, let $Y \subseteq C$ is given by Theorem 4 and put $R = R_A[C^{-1}]$. Then $|(A:A)Y| \le R|A^{-1}Y| \le R|A^{-1}C|$, $R = |AY^{-1}|/|Y| \le |AC^{-1}|/|C|$ and arguments similar to (16) can be applied. This completes the proof. $\Box$

Remark 10 *Actually, the proof of Lemma 9 gives us $d(AA) \le \frac{|AC|^4}{|AA||C|^3}$, $d(A:A) \le \frac{|AC|^4}{|A:A||C|^3}$ for any nonempty C.*

Finally, we formulate a full version of Theorem 1 from the introduction.

Theorem 11 *Let $A, B \subseteq \mathbb{R}$ be sets, $\tau > 0$ be a real number. Then*

$$|\{x \;:\; |A \cap xB| \ge \tau\}| \ll \frac{|A+A||B+B|}{\tau^2} \,. \tag{17}$$

In particular

$$\mathsf{E}^{\times}(A, B) \ll |A+A||B+B| \cdot \log(\min\{|A|, |B|\}) \,. \tag{18}$$

4 The proof of the main results

Our proof relies on a partial case of Theorem 14 from [15].

Theorem 12 *Suppose that $A, A_* \subset \mathbb{R}$ have SzT–type with the same parameter $\alpha = 2$. Then*

$$|A \pm A_*| \gg \max\{d(A_*)^{-\frac{1}{3}} d(A)^{-\frac{2}{9}} |A_*|^{\frac{8}{9}} |A|^{\frac{2}{3}}, d(A)^{-\frac{1}{3}} d(A_*)^{-\frac{2}{9}} |A|^{\frac{8}{9}} |A_*|^{\frac{2}{3}},$$

$$\min\{d(A_*)^{-\frac{2}{27}} d(A)^{-\frac{13}{27}} |A_*|^{\frac{14}{9}}, d(A)^{-\frac{2}{27}} d(A_*)^{-\frac{13}{27}} |A|^{\frac{14}{9}}\}\} \times (\log(|A||A_*|))^{-\frac{2}{9}}. \tag{19}$$

Now we can prove the main result of the paper.

Theorem 13 *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|A : A + A| \gg |A|^{\frac{3}{2} + \frac{1}{82}} \cdot (\log |A|)^{-\frac{2}{41}}, \tag{20}$$

and

$$|AA + A| \gg |AA|^{\frac{11}{41}} |A : A|^{-\frac{11}{41}} |A|^{\frac{62}{41}} (\log |A|)^{-\frac{2}{41}}. \tag{21}$$

Proof. Put $l = \log |A|$. Without loosing of generality, we can assume that $0 \notin A$. Suppose that $|A : A + A| \ll M|A|^{3/2}$, $|AA + A| \ll M|A|^{3/2}$, where M is a small power of $|A|$, that is $M = |A|^\varepsilon$ and obtain a contradiction. Let us begin with (20) because the proof of the second inequality requires some additional steps.

Recall the arguments from [1] or see the proof of Theorem 31 from [11]. Let l_i be the line $y = q_i x$. Thus, $(x, y) \in l_i \cap A^2$ if and only if $x \in A^\times_q$. Let $q_1, \dots, q_n \in \Pi \subseteq A/A$ be such that $q_1 < q_2 < \dots < q_n$. Here Π is a set which can vary, in principle, and at the moment we choose Π such that $|A^\times_{q_i}| \ge 2^{-1}|A|^2/|A/A|$ for all $q_i \in \Pi$. Thus, $\sum_{i\in\Pi} |A^\times_{q_i}| \ge \frac{1}{2}|A|^2$. We multiply all points of A^2 lying on the line l_i by $\Delta(A^{-1})$, so we obtain $|A^\times_{q_i} : A|$ points still belonging to the line l_i and then we consider sumset of the resulting set with $l_{i+1} \cap A^2$. Clearly, we get $|A^\times_{q_i} : A||A^\times_{q_{i+1}}|$ points from the set $(A : A + A)^2$ lying between the lines l_i and l_{i+1}. Therefore, using the definition of the number $d(A)$, we have

$$M^2|A|^3 \gg |A : A + A|^2 \ge \sum_{i=1}^{n-1} |A^\times_{q_i}||A^\times_{q_{i+1}} : A| \ge |A|^{1/2} d^{1/2}(A) \sum_{i=1}^{n-1} |A^\times_{q_i}|^{3/2} \gg \tag{22}$$

$$\gg |A|^{3/2} d^{1/2}(A) |A : A|^{-1/2} \sum_{i=1}^{n-1} |A^\times_{q_i}| \gg |A|^{7/2} d^{1/2}(A) |A : A|^{-1/2}. \tag{23}$$

Thus,

$$d(A) = \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|} \ll \frac{M^4 |A/A|}{|A|}. \tag{24}$$

To estimate $d(A:A)$, $d(AA)$ we use Lemma 9, see Remark 10. In other words, our C is some $A^\times_{q_i}$, where $q_i \in \Pi$. After that applying the first inequality of Theorem 12 with $A = A$, $A_* = A:A$, we obtain

$$M|A|^{3/2} \ge |A:A+A| \gg |A:A|^{8/9}|A|^{2/3}d^{-2/9}(A)\left(\frac{|A|^2d^2(A)}{|A:A||C|}\right)^{-1/3} l^{-2/9} =$$

$$= |A:A|^{11/9}d^{-8/9}(A)|C|^{1/3}l^{-2/9} \gg |A|^{14/9}M^{-32/9}l^{-2/9}\,,$$

and hence $M \gg l^{-2/41}|A|^{1/82}$. This implies (20).

It remains to prove (21). In the case we multiply all points of A^2 lying on the line l_i by $\Delta(A)$, so we obtain $|AA^\times_{q_i}|$ points still belonging to the line l_i and then we consider sumset of the resulting set with $l_{i+1} \cap A^2$. Clearly, we obtain $|AA^\times_{q_i}||A^\times_{q_{i+1}}|$ points from the set $(AA+A)^2$. Thus,

$$M^2|A|^3 \gg |AA+A|^2 \ge \sum_{i=1}^{n-1} |A^\times_{q_i}||AA^\times_{q_{i+1}}| \tag{25}$$

and we repeat the arguments above. The proof gives us

$$|AA+A| \gg |AA|^{11/41}|A|^{-4/41}(\mathsf{E}^\times_{3/2}(A))^{22/41}l^{-2/41}\,. \tag{26}$$

Here we have chosen the set Π as $\sum_{q\in\Pi}|A^\times_q|^{3/2} \gg \mathsf{E}^\times_{3/2}(A)$ or, in other words, $|A^\times_q| \gg (\mathsf{E}^\times_{3/2}(A))^2|A|^{-4}$. Using the Hölder inequality, combining with (26), we get

$$|AA+A| \gg |AA|^{\frac{11}{41}}|A:A|^{-\frac{11}{41}}|A|^{\frac{62}{41}}l^{-2/41}\,.$$

This completes the proof. $\Box$

Remark 14 *Using the full power of Theorem 14 from [15] one can obtain further results connecting $|AA:A|$, $|A:AA|$ with $|AA+A|$, $|A:A+A|$ and so on. We do not make such calculations.*

The same method allows us to improve the result of Balog concerning the size of $AA+AA$ and $A:A+A:A$.

Theorem 15 *Let $A \subset \mathbb{R}$ be a set. Then*

$$|A:A+A:A| \gg |A:A|^{\frac{14}{29}}|A|^{\frac{30}{29}}(\log|A|)^{-\frac{2}{29}}\,, \tag{27}$$

and

$$|AA+AA| \gg |AA|^{\frac{19}{29}}|A:A|^{-\frac{5}{29}}|A|^{\frac{30}{29}}(\log|A|)^{-\frac{2}{29}}\,. \tag{28}$$

P r o o f. As in the proof of Theorem 13, we define l_i to be the line $y = q_i x$ and $q_1, \dots, q_n \in \Pi \subseteq A/A$ be such that $q_1 < q_2 < \dots < q_n$ and $|A^\times_{q_i}| \ge 2^{-1}|A|^2/|A/A|$ for any $q_i \in \Pi$. Thus, $\sum_i |A^\times_{q_i}| \ge \frac{1}{2}|A|^2$. We multiply all points of A^2 lying on all lines l_i by $\Delta(A^{-1})$, so we obtain $|A^\times_{q_i} : A|$ points still belonging to the line l_i and then we consider sumset of the resulting set with itself. Clearly, we get $|A^\times_{q_i} : A||A^\times_{q_{i+1}} : A|$ points from the set $(A : A + A : A)^2$ lying between the lines l_i and l_{i+1}. Therefore, we have

$$\sigma^2 := |A : A + A : A|^2 \ge \sum_{i=1}^{n-1} |A^\times_{q_i} : A||A^\times_{q_{i+1}} : A| \ge$$

$$\ge d(A)|A| \sum_{i=1}^{n-1} |A^\times_{q_i}|^{1/2}|A^\times_{q_{i+1}}|^{1/2} \gg |A|^3 d(A) \,. \tag{29}$$

It gives $d(A) \ll \sigma^2 |A|^{-3}$. Using Theorem 12 with $A = A_* = A/A$, we obtain

$$\sigma \gg |A/A|^{14/9} \left(\frac{|A|^2 d^2(A)}{|A/A||C|} \right)^{-5/9} l^{-2/9} \gg |A/A|^{19/9} |A|^{-10/9} |C|^{5/9} d^{-10/9}(A) l^{-2/9} \gg \tag{30}$$

$$\gg |A/A|^{14/9} \sigma^{-20/9} |A|^{10/3} l^{-2/9} \,. \tag{31}$$

After some calculations, we get $\sigma \gg |A/A|^{14/29}|A|^{30/29} l^{-2/29}$.

To obtain (28) we use the previous arguments

$$\sigma^2 := |AA + AA|^2 \ge \sum_{i\in\Pi} |AA^\times_{q_i}||AA^\times_{q_{i+1}}| \ge d(A)|A| \sum_{i\in\Pi} |A^\times_{q_i}|^{1/2}|A^\times_{q_{i+1}}|^{1/2} \gg$$

$$\gg d(A)|A||\Pi|\Delta \,, \tag{32}$$

choosing $\Pi \subseteq A/A$ such that for any $q \in \Pi$ one has $|A|^2/|A : A| \ll \Delta \le |A^\times_q|$. Clearly, such set Π exists by simple average arguments. The calculations as in (30)—(31) give us

$$\sigma \gg |AA|^{14/9} \left(\frac{|A|^2 d^2(A)}{|AA|\Delta} \right)^{-5/9} l^{-2/9} \gg |AA|^{19/9} |A|^{-10/9} \Delta^{5/9} d^{-10/9}(A) l^{-2/9} \gg$$

$$\gg |AA|^{19/9} \sigma^{-20/9} (|\Pi|\Delta^{3/2})^{10/9} l^{-2/9} \,.$$

After some computations, we obtain

$$\sigma \gg |AA|^{19/29} |A : A|^{-5/29} |A|^{30/29} l^{-2/29} \,.$$

This concludes the proof. $\square$

Finally, let us obtain a result on $AA + A$, $AA + AA$ of another type.

Proposition 16 *Let $A \subset \mathbb{R}$ be a set. Then*

$$|AA+A|^4\,,\ |A:A+A|^4 \gg |A|^{-2} (\mathsf{E}^\times_{3/2}(A))^2 \mathsf{E}^+_3(A) \log^{-1} |A|\,, \tag{33}$$

and

$$|AA+AA|^2\,,\ |A:A+A:A|^2 \gg \mathsf{E}^+_3(A) \log^{-1} |A|\,. \tag{34}$$

Moreover

$$|AA+A|^4\,,\ |A:A+A|^4 \gg \frac{|A|^{10}}{|A:A||A-A|^2}\,, \tag{35}$$

$$|AA+AA|^2\,,\ |A:A+A:A|^2 \gg \frac{|A|^{6}}{|A-A|^2}\,. \tag{36}$$

P r o o f. Put $l = \log|A|$. Using Lemma 7, we obtain for any A, B and C

$$\sum_x (A\circ A)(x)(B\circ B)(x)(C\circ C)(x) \ll |A||B||C| (d(A)d(B)d(C))^{1/3} \log(|A||B||C|)\,. \tag{37}$$

In particular case $A=B=C$ of formula above the definition of the number $d(A)$ gives us

$$|AA^\times_s|^2\,,\ |A:A^\times_s|^2 \gg |A|^{-2} |A^\times_s| \mathsf{E}^+_3(A) l^{-1} \tag{38}$$

for any $s\in A:A$. Applying (22), (25) and the last bound, we obtain (33). Using (38) one more time and Katz–Koester inclusion [3], namely,

$$AA^\times_s \subseteq AA \cap sAA\,, \quad A:A^\times_s \subseteq (A:A) \cap s^{-1} (A:A) \tag{39}$$

as well as formula (17) of Solymosi's result, we get (34). Another way to prove (34) is just to use formulas (29), (32), combining with (38).

Inequalities (35), (36) follow similar to (33), (34) from a direct application of Definition 6 and the Hölder inequality. This completes the proof. $\Box$

Remark 17 *Applying arguments as in the proof (34) as well as formula (12) of Lemma 7, we obtain a similar bound, namely,*

$$\mathsf{E}^+(A) \ll |A||AA+AA|$$

(actually, using methods from [13] one can improve the inequality). It is interesting to compare this estimate with Solymosi's upper bound for the multiplicative energy (18). Using formula (11) of Lemma 7, we have also

$$(\mathsf{E}^+(A))^{3/2} \mathsf{E}^\times_{3/2}(A) \ll \mathsf{E}^+_{3/2}(A) |A||AA+A|^2\,.$$

Combining inequality (34) with some estimates from [16], we obtain a result in spirit of paper [9].

Corollary 18 *Let $A \subset \mathbb{R}$ be a set. Suppose that*

$$|(A+A)(A+A)+(A+A)(A+A)| \ll |A|^2 \quad \textit{and} \quad \mathsf{E}^{+}(A)|A-A| \ll |A|^4 \,. \tag{40}$$

Then

$$|A-A| \ll |A| \log^{4/7} |A| \,. \tag{41}$$

The same holds if one replace sum onto minus and product onto division into the first condition from (40).

If just the first condition of (40) holds (with plus) then

$$|A \pm A| \ll |A| \log |A| \,, \tag{42}$$

and if it holds with minus then

$$|A - A| \ll |A| \log |A| \,, \tag{43}$$

Again, one can replace product onto division in the first condition of (40).

Proof. Let us have deal with the situation of the sum and the product. Another cases can be considered similarly. By Theorem 30 from [16] and our second condition, one has

$$\mathsf{E}^{+}_3(A \pm A) \ge |A|^{45/4}|A-A|^{-1/2}(\mathsf{E}^{+}(A))^{-9/4} \gg |A|^{9/4}|A-A|^{7/4} \,.$$

On the other hand, using formula (34) from Proposition 16 and our first condition, we get

$$|A|^4 \log |A| \gg \mathsf{E}^{+}_3(A \pm A) \gg |A|^{9/4}|A-A|^{7/4}$$

as required.

Finally, using the additive variant of Katz–Koester inclusion (39) (or see Proposition 29 from [16]), we obtain

$$|A|^3|A \pm A| \le \mathsf{E}^{+}_3(A+A) \ll |A|^4 \log |A| \,,$$

and

$$|A|^3|A - A| \le \mathsf{E}^{+}_3(A-A) \ll |A|^4 \log |A| \,.$$

This completes the proof. $\Box$

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